

BOHR TOPOLOGY AND DIFFERENCE SETS FOR SOME ABELIAN GROUPS

JOHN T. GRIESMER

ABSTRACT. For a prime number p , let G_p be the countable direct sum of \mathbb{F}_p , where \mathbb{F}_p is the field of p elements. Viewing G_p as a countable abelian group, we construct sets $A \subseteq G_p$ having upper Banach density close to $\frac{1}{2}$ (if $p = 2$) or $\frac{1}{2} - \frac{1}{2p}$ (if p is odd), such that $A - A := \{a - b : a, b \in A\}$ does not contain a Bohr neighborhood of any $g \in G_p$. This construction answers a variant of a question asked by several authors: if A is a set of integers having positive upper Banach density, must $A - A$ contain a Bohr neighborhood of some $n \in \mathbb{Z}$?

Our construction also answers a variant of a question posed by the author by providing examples of sets $A, S \subseteq G_p$ such that A has positive upper Banach density and S is dense in the Bohr topology of G_p , while $A + S$ is not piecewise Bohr.

1. INTRODUCTION

1.1. Density, difference sets, and Bohr neighborhoods. The following question has been asked in several articles, including [2], [6], [7], and [9]. Here d^* denotes the upper Banach density and a Bohr neighborhood is an open set in the Bohr topology of \mathbb{Z} ; see Sections 2.5 and 2.6 for definitions.

Question 1.1. If A is a set of integers having $d^*(A) > 0$, must $A - A := \{a - b : a, b \in A\}$ contain a Bohr neighborhood of some $n \in \mathbb{Z}$?

We do not answer Question 1.1. Instead we answer the analogous question for some countable abelian groups besides \mathbb{Z} . If p is a prime number, let \mathbb{F}_p denote the field with p elements. Let G_p be the countable direct sum $\bigoplus_{n=1}^{\infty} \mathbb{F}_p$, with the group operation of coordinatewise addition.

Theorem 1.2. *Let p be a prime number and let $\delta_2 = \frac{1}{2}$, $\delta_p = \frac{1}{2} - \frac{1}{2p}$ for odd p .*

Date: January 4, 2017.

1. For all $\varepsilon > 0$, there is a set $A \subseteq G_p$ satisfying $d^*(A) > \delta_p - \varepsilon$ such that $A - A$ does not contain a Bohr neighborhood.
2. If $A \subseteq G_p$ has $d^*(A) > \delta_p$, then $A - A = G_p$.

For the groups G_p , a Bohr neighborhood is simply a set containing a coset of a finite index subgroup of G_p .

The techniques we use to prove Theorem 1.2 do not seem to apply to groups with few finite index subgroups, such as \mathbb{Z} and \mathbb{Q} . The obstruction seems to be the lack of independent subsets¹: while most k -tuples of elements of G_p are independent, no pair of nonzero elements of \mathbb{Z} is independent.

1.2. Sumsets. The following question is closely related to Question 1.1. See Section 2.7 for the definition of “piecewise Bohr,” and see [11] for a definition of the Bohr compactification.

Question 1.3 (cf. Question 5.1 in [7]). Let G be a countable abelian group, $S \subseteq G$ and let \tilde{S} denote the closure of S in bG , the Bohr compactification of G . Let m_{bG} denote Haar measure on bG . Which, if any, of the following implications are true?

- (1) If $m_{bG}(\tilde{S}) > 0$ and $d^*(A) > 0$, then $A + S$ is piecewise syndetic.²
- (2) If $m_{bG}(\tilde{S}) > 0$ and $d^*(A) > 0$, then $A + S$ is piecewise Bohr.
- (3) If S is dense in the Bohr topology of G and $d^*(A) > 0$, then $d^*(A + S) = 1$.

The sets we construct to prove Theorem 1.2 are counterexamples to the implications (2) and (3) of Question 1.3 for $G = G_p$, resulting in the following theorem. When $G = G_2$ none of the implications (1)-(3) hold – see [8] for details.

Theorem 1.4. For all primes p and $\varepsilon > 0$, there are sets $A, S \subseteq G_p$ such that $d^*(A) > \delta_p - \varepsilon$, S is dense in the Bohr topology of G_p , and $A + S$ is not piecewise Bohr.

If $d^*(A + S) = 1$ then $A + S$ is piecewise Bohr, so Theorem 1.4 implies that both (2) and (3) of Question 1.3 fail for $G = G_p$.

¹A finite subset S of an abelian group G is *independent* if $\sum_{s \in S} n_s s = 0_G$, $n_s \in \mathbb{Z}$, implies $n_s s = 0_G$ for all $s \in S$.

²We will not use syndeticity or piecewise syndeticity in this article. See [8] for more on Part (1) of Question 1.3.

1.3. Context. The following classical result is due to Bogoliouboff [3] for $G = \mathbb{Z}$ and Følner [4] for countable abelian groups G .

Theorem 1.5. *If G is a countable abelian group and $A \subseteq G$ has $d^*(A) > 0$, then*

1. $A - A$ contains a set of the form $B \setminus Z$, where B is a Bohr neighborhood of 0 and $d^*(Z) = 0$.
2. $(A - A) - (A - A)$ contains a Bohr neighborhood of 0.

Naturally one wonders whether the set Z can be eliminated in the statement of Theorem 1.5. When $G = \mathbb{Z}$, the main result of [10] implies that it cannot: for all $\varepsilon > 0$, there is a set of integers A having $d^*(A) > \frac{1}{2} - \varepsilon$, and $A - A$ does not contain a Bohr neighborhood of 0. An analogous construction for $G = G_2$ is provided in [5], showing that the difference set $A - A$ may not contain a Bohr neighborhood of 0 when $A \subseteq G_2$ has $d^*(A) > 0$. The constructions in [5, 10] do not eliminate the possibility that $A - A$ contains a Bohr neighborhood of *some* $n \in \mathbb{Z}$ (or $g \in G_2$), so Question 1.1 remains open, while Theorem 1.2 resolves the analogous question for G_p and suggests a negative answer to Question 1.1.

Question 1.1 and its generalization to countable abelian groups can be phrased in terms of recurrence for measure preserving actions. See [8] for exposition and many related problems.

1.4. Outline. Section 2 introduces a useful presentation of G_p and proves some facts about the Bohr topology of G_p . Some sets are shown to be dense in the Bohr topology of G_p . Section 3 covers the construction of the sets A for Part 1 of Theorem 1.2. Section 4 contains the proofs of Theorems 1.2 and 1.4.

The proof of Part 1 of Theorem 1.2 constructs two subsets of G_p , A and S , where $d^*(A) > \delta_p - \varepsilon$ and S is dense in the Bohr topology of G_p , such that $(A - A) \cap S = \emptyset$. It follows that $A - A$ does not contain a Bohr neighborhood, since every Bohr neighborhood has nontrivial intersection with V (a dense set).

A simpler proof, and stronger conclusions, for the special case of Theorems 1.2 and 1.4 where $p = 2$ appears in [8].

2. VECTOR SPACES OVER \mathbb{F}_p AND THEIR BOHR TOPOLOGIES

If G is an abelian group and $A, B \subseteq G$, $g \in G$, we write $A + B$ for $\{a + b : a \in A, b \in B\}$, $A - B$ for $\{a - b : a \in A, b \in B\}$, and $A + g$ for $\{a + g : a \in A\}$.

If p is a prime number, let \mathbb{F}_p denote the finite field with p elements. We write the elements of \mathbb{F}_p as $0, 1, \dots, p-1$. Consider the countable direct sum $G_p := \bigoplus_{n=1}^{\infty} \mathbb{F}_p$, where p is a fixed prime.

2.1. Presentation of G_p . Let $\Omega := \{0, 1\}^{\mathbb{N}}$, and write elements of Ω as $\omega = \omega_1 \omega_2 \omega_3 \dots$. For each $n \in \mathbb{N}$ let $\Omega_n = \{0, 1\}^{\{1, \dots, n\}}$ and $\pi_n : \Omega \rightarrow \Omega_n$ be the projection map given by $\pi_n(\omega) = \omega_1 \dots \omega_n$. Let Γ_p be the group of functions $g : \Omega \rightarrow \mathbb{F}_p$ with the group operation of pointwise addition.

For each $n \in \mathbb{N}$, let $G_p^{(n)}$ be the subgroup of Γ_p consisting of functions of the form $f \circ \pi_n$ where $f : \Omega_n \rightarrow \mathbb{F}_p$. Observing that $G_p^{(n)} \subseteq G_p^{(n+1)}$ for each n , we let $\tilde{G}_p := \bigcup_{n \in \mathbb{N}} G_p^{(n)}$. Then \tilde{G}_p is a countable abelian group isomorphic³ to G_p . Our constructions are easier to work with from the perspective of \tilde{G}_p , rather than the standard presentation of a countable direct sum, so from now on we let G_p denote the group \tilde{G}_p .

We observe that $G_p^{(n)}$ is isomorphic to $(\mathbb{F}_p)^{\Omega_n}$, and we will identify elements of $G_p^{(n)}$ with elements of $(\mathbb{F}_p)^{\Omega_n}$. The identification is given by $g \leftrightarrow \tilde{g}$ if and only if $g = \tilde{g} \circ \pi_n$ for $\tilde{g} \in (\mathbb{F}_p)^{\Omega_n}$.

Let $G_p^{(0)}$ denote the group of constant functions $f : \Omega \rightarrow \mathbb{F}_p$, so that $G_p^{(0)} \subseteq G_p^{(1)}$. Let $\mathbf{1} \in G_p$ denote the constant function where $\mathbf{1}(\omega) = 1 \in \mathbb{F}_p$ for every $\omega \in \Omega$. For $x \in \mathbb{F}_p$, define $x\mathbf{1}$ to be the constant function having $x\mathbf{1}(\omega) = x$ for all $\omega \in \Omega$, and let $\mathbf{0}$ denote $0\mathbf{1}$, the identity element of G_p .

Remark 2.1. The group G_p is the group of continuous functions $g : \Omega \rightarrow \mathbb{F}_p$, where Ω has the product topology and \mathbb{F}_p has the discrete topology. Another description of G_p is that it is the group of functions $g : \Omega \rightarrow \mathbb{F}_p$ where $g(\omega)$ depends on only finitely many coordinates of ω .

2.2. Cylinder sets. If $\tau \in \Omega_n$, let $[\tau] \subseteq \Omega$ be $\pi_n^{-1}(\tau)$, so that $[\tau] := \{\omega \in \Omega : \omega_i = \tau_i \text{ for } 1 \leq i \leq n\}$. We call $[\tau]$ a *cylinder set*. Each cylinder set $[\tau]$ is homeomorphic to Ω by the map $\theta : [\tau] \rightarrow \Omega$, $\theta(\omega) = \omega_{n+1}\omega_{n+2}\dots$.

Observe that $G_p^{(n)}$ is the group of functions $g : \Omega \rightarrow \mathbb{F}_p$ which are constant on the cylinder sets $[\tau]$, where $\tau \in \Omega_n$.

Definition 2.2. If $E \subseteq \Omega$, let $|E|_n := |\{\tau \in \Omega_n : [\tau] \subseteq E\}|$.

³One can construct the isomorphism by hand, but when p is prime it suffices to observe that both G_p and \tilde{G}_p are countably infinite vector spaces over the finite field \mathbb{F}_p .

The above definition will usually be applied to sets of the form $g^{-1}(S)$ where $g \in G_p^{(n)}$ and $S \subseteq \mathbb{F}_p$. We list some relevant properties.

Observation 2.3. (C1) $|E|_n \leq 2^n$ for all $E \subseteq \Omega$.

(C2) For an element $g \in G_p^{(n)}$, if $g = \tilde{g} \circ \pi_n$, then $|g^{-1}(1)|_n = |\tilde{g}^{-1}(1)|$.

(C3) If $g \in G_p^{(n)}$, $A, B \subseteq \mathbb{F}_p$, and $A \cap B = \emptyset$, then

$$|g^{-1}(A) \cup g^{-1}(B)|_n = |g^{-1}(A)|_n + |g^{-1}(B)|_n.$$

2.3. Restrictions to cylinders. Given $m, n \in \mathbb{N}$ with $m < n$, a string $\tau \in \Omega_m$, and an element $g \in G_p^{(n)}$, we define $g|_\tau$ to be the element of $G_p^{(n-m)}$ satisfying $g|_\tau(\omega) = g(\tau\omega)$ for all $\omega \in \{0, 1\}^{(n-m)}$, where $\tau\omega \in \Omega_n$ is the concatenation of τ and ω . To give an explicit example: for $n = 5$, $m = 2$, and $g : \Omega_5 \rightarrow \mathbb{F}_7$, if $\tau = 01$, then $g|_\tau(011) = g(01011)$. With this notation we can identify $g \in G_p^{(n)}$ with the function $f : \Omega_n \rightarrow G_p^{(n-m)}$, where $f(\tau) := g|_\tau$ for each $\tau \in \Omega_m$. This identification is used in Section 3.4, where the sets A of Theorems 1.2 and 1.4 are defined.

2.4. Hamming Balls. Let $U(n, k)$ be the set of $g \in G_p^{(n)}$ satisfying $|\{\omega \in \Omega : g(\omega) \neq 0\}|_n \leq k$. This is the *Hamming ball of scale n and radius k* around $0 \in G_p^{(n)}$. In other words, $U(n, k)$ is the set of $g \in G_p$ which are constant on the cylinder sets $[\tau]$ for $\tau \in \Omega_n$ and $g|_\tau = \mathbf{0}$ for at least $|\Omega_n| - k$ such τ .

For $g \in G_p$, let $V(n, k) := U(n, k) + \mathbf{1}$, so that

$$V(n, k) = \{g \in G_p^{(n)} : |\{\omega \in \Omega : g(\omega) \neq 1\}|_n \leq k\}.$$

The rough idea of the proof of Theorem 1.2 is that Hamming balls $V(n, k)$ with sufficiently large radius k meet all cosets of many finite index subgroups of G_p , while there are large subsets A of $G_p^{(n)}$ such that $(A - A) \cap V(n, k) = \emptyset$.

Remark 2.4. We call the sets $U(n, k)$ and $V(n, k)$ “Hamming balls” as we may identify elements of $G_p^{(n)}$ with strings of length 2^n from the alphabet \mathbb{F}_p . With this identification $U(n, k)$ is the set of strings differing from the constant 0 string in at most k coordinates.

2.5. Bohr topology of G_p . Here \mathbb{T} denotes the group \mathbb{R}/\mathbb{Z} with the usual topology.

The Bohr topology of a (discrete) abelian group G is the weakest topology such that every homomorphism $\rho : G \rightarrow \mathbb{T}$ is continuous. Since every element of G_p has order p , the image of every homomorphism $\rho : G_p \rightarrow \mathbb{T}$ is contained in the finite set of elements of \mathbb{T} having

order p . Consequently, a base for the Bohr topology of G_p consists of the cosets of finite-index subgroups of G_p . This description is a proof of the following characterization of denseness in the Bohr topology.

Lemma 2.5. *A set $S \subseteq G_p$ is dense in the Bohr topology of G_p iff for every homomorphism $\rho : G_p \rightarrow K$, where K is a finite group, $\rho(S) = \rho(G_p)$.*

With this characterization we can prove that unions of certain Hamming balls are dense in the Bohr topology of G_p .

Lemma 2.6. *Let $(g_i)_{i \in \mathbb{N}}$ be a sequence of elements of G_p , and let $(n_i)_{i \in \mathbb{N}}$, $(k_i)_{i \in \mathbb{N}}$ be sequences of natural numbers where $n_i \rightarrow \infty$ and $k_i \rightarrow \infty$. Then $S := \bigcup_{i=1}^{\infty} U(n_i, k_i) + g_i$ is dense in the Bohr topology of G_p .*

Proof. By Lemma 2.5 it suffices to show that for every homomorphism ρ to a finite group K $\rho(U(n_i, k_i) + g_i) = \rho(G_p)$ for some i . Without loss of generality we may assume ρ is surjective, and then it suffices to show that

$$(2.1) \quad \rho(U(n_i, k_i) + g_i) = K \text{ for some } i.$$

Fixing such a homomorphism ρ , we aim to establish the assertion (2.1).

Let $d, n \in \mathbb{N}$, and consider the following elementary facts.

- (i) Since G_p is a vector space over \mathbb{F}_p , every subgroup $H \leq G_p$ is a subspace, and the quotient map $G_p \rightarrow G_p/H$ is linear.
- (ii) If $F \subseteq (\mathbb{F}_p)^d =: V_d$ is invariant under scalar multiplication and generates V_d as an abelian group, then every element of V_d is the sum of at most d elements of F . (Here we consider V_d as a vector space over \mathbb{F}_p with the usual operations.)
- (iii) $U(n, 1)$ generates $G_p^{(n)}$ as an abelian group.
- (iv) $U(n, 1)$ is invariant under scalar multiplication: $cf \in U(n, 1)$ for every $f \in U(n, 1)$ and $c \in \mathbb{F}_p$.
- (v) $U(n, d)$ contains all sums of at most d elements of $U(n, 1)$.

In light of fact (i), we can assume that the image of ρ is $(\mathbb{F}_p)^d$ for some $d \in \mathbb{N}$.

Choose n sufficiently large that $\rho(G_p^{(n)}) = K$. If $n_i \geq n$, then $G_p^{(n)} \subseteq G_p^{(n_i)}$, and $U(n_i, 1)$ generates $G_p^{(n_i)}$. Then $\rho(U(n_i, 1))$ generates K , so if $k_i \geq d$, then every element of K is a sum of at most k_i elements of $\rho(U(n_i, 1))$, and $\rho(U(n_i, k_i))$ contains all such sums. Thus $\rho(U(n_i, k_i)) = K$, and we have established the assertion (2.1). \square

2.6. Upper Banach Density. If G is an abelian group, a *Følner sequence* for G is a sequence of finite subsets $\Phi_n \subseteq G$ such that $\lim_{n \rightarrow \infty} \frac{|(\Phi_n + g) \triangle \Phi_n|}{|\Phi_n|} = 0$ for all $g \in G$. If $\Phi = (\Phi_n)_{n \in \mathbb{N}}$ is a Følner sequence and $A \subseteq G$, the *upper density of A with respect to Φ* is $\bar{d}_\Phi(A) := \limsup_{n \rightarrow \infty} \frac{|A \cap \Phi_n|}{|\Phi_n|}$. The *upper Banach density* of the set A is $d^*(A) := \sup\{\bar{d}_\Phi(A) : \Phi \text{ is a Følner sequence}\}$.

When $G = G_p$, the sequence of subgroups $G_p^{(n)}$ forms a Følner sequence, so

$$(2.2) \quad d^*(A) \geq \limsup_{n \rightarrow \infty} \frac{|A \cap G_p^{(n)}|}{|G_p^{(n)}|}.$$

2.7. Piecewise Bohr sets. Let G be a countable abelian group.

Definition 2.7. A set $S \subseteq G$ is *piecewise Bohr* if there is a nonempty Bohr neighborhood $B \subseteq G$ such that for every finite $F \subseteq B$, there exists $g \in G$ such that $g + F \subseteq S$.

See [7] for a proof that the above definition of piecewise Bohr is equivalent to the original definition given in [1] (in the case $G = \mathbb{Z}$).

Observation 2.8. If $C \subseteq G$ is piecewise Bohr, then $C - C$ contains a Bohr neighborhood of 0, since there is a Bohr neighborhood B such that $C - C$ contains $F - F$ for every finite $F \subseteq B$.

3. CONSTRUCTION OF SOME DENSE SETS

In this section we define the sets A described in Theorems 1.2 and 1.4. The set A will be a union of inductively defined sets. The base case is defined in Section 3.3, and the induction step is defined in Section 3.4. Sections 3.1 and 3.2 present some preliminary definitions and lemmas.

3.1. Bias patterns. We maintain the notation and conventions of Section 2 and Definition 2.2.

Let Y be a set and $k, n \in \mathbb{N}$. If $\mathcal{P} = (Y_0, \dots, Y_{k-1}, Z)$ is an ordered partition of Y into $k + 1$ sets and $S \subseteq \{0, \dots, k - 1\}$, let $\text{Bias}_n(Y, \mathcal{P}, S, m)$ be the set of functions $f : \Omega \rightarrow Y$ satisfying the following conditions:

- (i) f is constant on the cylinder sets $[\tau]$ where $\tau \in \Omega_n$,
- (ii) $f(X) \subseteq \bigcup_{i=0}^{k-1} Y_i$,
- (iii) $|f^{-1}(Y_i)|_n > \frac{1}{k}|\Omega_n| + m$ for all $i \in S$,
- (iv) $|f^{-1}(Y_i)|_n < \frac{1}{k}|\Omega_n| - m$ for all $i \notin S$.

We allow Z to be the empty set.

Lemma 3.1. *Let X and Y be finite sets, with $\mathcal{P} = (Y_0, \dots, Y_{k-1}, Z)$ an ordered partition of Y , and $n, k \in \mathbb{N}, m \in \mathbb{N} \cup \{0\}$. Then*

- (B1) $\text{Bias}_n(Y, \mathcal{P}, \{0, \dots, k-1\}, m) = \emptyset$.
- (B2) $\text{Bias}_n(Y, \mathcal{P}, \emptyset, m) = \emptyset$.
- (B3) *If $S \neq S' \subseteq \{0, \dots, k-1\}$, then*

$$\text{Bias}_n(Y, \mathcal{P}, S, m) \cap \text{Bias}_n(Y, \mathcal{P}, S', m) = \emptyset.$$

Proof. (B1): For $f \in \text{Bias}_n(Y, \mathcal{P}, \{0, \dots, k-1\}, m)$, we have

$$\sum_{i=0}^{k-1} |f^{-1}(Y_i)|_n > |\Omega_n| + km,$$

which implies $|\Omega_n| > |\Omega_n| + km$, a contradiction. (B2) follows similarly.

(B3): Let $f \in \text{Bias}_n(Y, \mathcal{P}, S, m) \cap \text{Bias}_n(Y, \mathcal{P}, S', m)$, and let $i \in S \triangle S'$. Then $|f^{-1}(Y_i)|_n > \frac{1}{k}|\Omega_n| + m$ and $|f^{-1}(Y_i)|_n < \frac{1}{k}|\Omega_n| - m$, a contradiction. \square

3.2. Orbits in $\{0, 1\}^{\mathbb{F}_p}$ by translation.

Lemma 3.2. *Let p be prime, let $\mathcal{S} = \{S \subset \mathbb{F}_p : S \neq \emptyset, \mathbb{F}_p\}$, and let \mathbb{F}_p (as an additive group) act on \mathcal{S} by translation: for $x \in \mathbb{F}_p$, $S \subseteq \mathbb{F}_p$, $T^x S := \{s + x : s \in S\}$. For all $S \in \mathcal{S}$, the orbit of S under T has cardinality p .*

Proof. If $S \subseteq \mathbb{F}_p$ is nonempty and $S \neq \mathbb{F}_p$, then $T^1 S \neq S$, so the orbit of S has cardinality > 1 . Since \mathbb{F}_p has prime order p , the T -orbits have cardinality 1 or cardinality p . \square

3.3. Partition of $G_p^{(n)}$. Fix a prime number p . We now construct partitions of the groups $G_p^{(n)}$. For each increasing sequence of natural numbers n_1, \dots, n_l and each sequence of natural numbers m_1, \dots, m_l , we will define a partition $\mathcal{P}^{(n_1, m_1), \dots, (n_l, m_l)}$ of $G_p^{(n_l)}$. For $l > 1$, the definition of $\mathcal{P}^{(n_1, m_1), \dots, (n_l, m_l)}$ will depend on $\mathcal{P}^{(n_2 - n_1, m_2), (n_3 - n_1, m_2), \dots, (n_l - n_1, m_l)}$.

We define $\mathcal{P}^{(n, m)}$ in this subsection and we define $\mathcal{P}^{(n_1, m_1), \dots, (m_l, n_l)}$, $l > 1$, in the next.

Let $\mathcal{S} = \{S \subset \mathbb{F}_p : S \neq \emptyset, \mathbb{F}_p\}$. Applying Lemma 3.2, fix a partition of \mathcal{S} into p disjoint nonempty collections $\mathcal{S}_x, x \in \mathbb{F}_p$, such that

- $\mathcal{S}_x + y = \mathcal{S}_{x+y}$, and
- $\{x\} \in \mathcal{S}_x$,

for all $x, y \in \mathbb{F}_p$. Here $\mathcal{S}_x + y$ denotes $\{S + y : S \in \mathcal{S}_x\}$. For example, with $p = 5$ we can take

$$\begin{aligned}\mathcal{S}_0 &= \{\{0\}, \{0, 1\}, \{0, 2\}, \{0, 1, 2\}, \{0, 2, 3\}, \{0, 1, 2, 3\}\} \\ \mathcal{S}_1 &= \{\{1\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}, \{1, 3, 4\}, \{1, 2, 3, 4\}\} \\ \mathcal{S}_2 &= \{\{2\}, \{2, 3\}, \{2, 4\}, \{2, 3, 4\}, \{2, 4, 0\}, \{2, 3, 4, 0\}\} \\ \mathcal{S}_3 &= \{\{3\}, \{3, 4\}, \{3, 0\}, \{3, 4, 0\}, \{3, 0, 1\}, \{3, 4, 0, 1\}\} \\ \mathcal{S}_4 &= \{\{4\}, \{4, 0\}, \{4, 1\}, \{4, 0, 1\}, \{4, 1, 2\}, \{4, 0, 1, 2\}\}.\end{aligned}$$

Definition 3.3. For $n, m \in \mathbb{N}$, let $\mathcal{P}^{(n,m)} = (P_0^{(n,m)}, \dots, P_{p-1}^{(n,m)}, Z^{(n,m)})$, where

$$\begin{aligned}P_x^{(n,m)} &:= \bigcup_{S \in \mathcal{S}_x} \text{Bias}_n(\mathbb{F}_p, (\{0\}, \dots, \{p-1\}, \emptyset), S, m), \\ Z^{(n,m)} &:= G_p^{(n)} \setminus \bigcup_{x \in \mathbb{F}_p} P_x^{(n,m)}.\end{aligned}$$

For example, with $p = 3$, we may take $\mathcal{S}_0 = \{\{0\}, \{0, 1\}\}$, $\mathcal{S}_1 = \{\{1\}, \{1, 2\}\}$, $\mathcal{S}_2 = \{\{2\}, \{2, 0\}\}$. We have

$$\begin{aligned}B_0 &:= \text{Bias}_n(\mathbb{F}_3, (\{0\}, \{1\}, \{2\}, \emptyset), \{0\}, m) \\ &= \{g \in G_3^{(n)} : |g^{-1}(0)|_n > \tfrac{1}{3}|\Omega_n| + m, |g^{-1}(i)|_n < \tfrac{1}{3}|\Omega_n| - m, i = 1, 2\}, \\ B_1 &:= \text{Bias}_n(\mathbb{F}_3, (\{0\}, \{1\}, \{2\}, \emptyset), \{0, 1\}, m) \\ &= \{g \in G_3^{(n)} : |g^{-1}(i)|_n > \tfrac{1}{3}|\Omega_n| + m, i = 0, 1, |g^{-1}(2)|_n < \tfrac{1}{3}|\Omega_n| - m\},\end{aligned}$$

and $P_0^{(n,m)} = B_0 \cup B_1$.

The fact that $\mathcal{P}^{(n,m)}$ is a partition of $G_p^{(n)}$ follows from Lemma 3.1.

Remark 3.4. The motivation for Definition 3.3 is to find a partition of $G_p^{(n)}$ into $p + 1$ cells P_0, \dots, P_{p-1}, Z satisfying the following for each $x \in \mathbb{F}_p$:

- (i) $(P_x - P_x) \cap V(n, m) = \emptyset$ when n is much larger than m .
- (ii) $|P_x| \approx \frac{1}{p}|G_p^{(n)}|$.

When $p = 2$, such a partition is constructed quite easily (see Definition 4.5 of [8]) by considering those functions $g \in G_2^{(n)}$ where $|g^{-1}(1)|_n$ is slightly greater than or slightly less than the expected value of $\frac{1}{2}|\Omega_n|$.

When $p > 2$, it is natural to partition $G_p^{(n)}$ based on how the values $|g^{-1}(x)|_n$ deviate from the expected value $\frac{1}{p}|\Omega_n|$, leading to the sets $\text{Bias}_n(\mathbb{F}_p, (\{0\}, \dots, \{p-1\}, \emptyset), S, m)$. The goal of satisfying property (i) above leads to the partition of $\{S \subset \mathbb{F}_p : S \neq \emptyset, \mathbb{F}_p\}$ and Definition 3.3.

For the next lemma recall the definition of $x\mathbf{1}$ from Section 2.1.

Lemma 3.5. *For all $x, y \in \mathbb{F}_p$, all $n, m, k \in \mathbb{N}$, where $k < m$ and $u \in U(n, k)$,*

- (i) $P_x^{(n,m)} + y\mathbf{1} = P_{x+y}^{(n,m)}$,
- (ii) $P_x^{(n,m)} + u \subseteq P_x^{(n,m-k)}$,
- (iii) $P_x^{(n,m)} + u + y\mathbf{1} \subseteq P_{x+y}^{(n,m-k)}$,
- (iv) If $x \neq y$, then $P_x^{(n,m)} \cap P_y^{(n,m)} = \emptyset$,
- (v) $P_x^{(n,m)} \subseteq P_x^{(n,m-k)}$.
- (vi) If $x \neq y$ then $P_x^{(n,m)} \cap P_y^{(n,m-k)} = \emptyset$,
- (vii) $x\mathbf{1} \in P_x^{(n,m)}$.

Note: Part (i) implies $|P_x^{(n,m)}| = |P_y^{(n,m)}|$ for all $x, y \in \mathbb{F}_p$.

Proof. (i) For all $S \subseteq \mathbb{F}_p$, we have

$$\begin{aligned} \text{Bias}_n(\mathbb{F}_p, (\{0\}, \dots, \{p-1\}, \emptyset), S, m) + y\mathbf{1} \\ = \text{Bias}_n(\mathbb{F}_p, (\{0\}, \dots, \{p-1\}, \emptyset), S + y, m). \end{aligned}$$

Together with the fact that $\mathcal{S}_x + y = \mathcal{S}_{x+y}$ and the definition of $P_x^{(n,m)}$, we conclude that $P_x^{(n,m)} + y\mathbf{1} = P_{x+y}^{(n,m)}$.

(ii) If $g \in P_x^{(n,m)}$, then $g \in \text{Bias}_n(\mathbb{F}_p, (\{0\}, \dots, \{p-1\}, \emptyset), S, m)$ for some $S \in \mathcal{S}_x$. Since $g([\tau]) + u([\tau]) = g([\tau])$ for all but k values of $\tau \in \Omega_n$, we have $g + u \in \text{Bias}_n(\mathbb{F}_p, (\{0\}, \dots, \{p-1\}, \emptyset), S, m - k)$. Hence, $g + u \in P_x^{(n,m-k)}$.

Part (iii) follows from parts (i) and (ii).

(iv) Lemma 3.1 implies that when $S \neq S' \subseteq \mathbb{F}_p$, the sets

$$\begin{aligned} B &:= \text{Bias}_n(\mathbb{F}_p, (\{0\}, \dots, \{p-1\}, \emptyset), S, m_1), \\ B' &:= \text{Bias}_n(\mathbb{F}_p, (\{0\}, \dots, \{p-1\}, \emptyset), S', m_1) \end{aligned}$$

are disjoint, which implies the conclusion of Part (iv).

(v) It suffices to show that for each $S \subseteq \mathbb{F}_p$, the set

$$B_1 := \text{Bias}_n(\mathbb{F}_p, (\{0\}, \dots, \{p-1\}, \emptyset), S, m)$$

is contained in $B'_1 := \text{Bias}_n(\mathbb{F}_p, (\{0\}, \dots, \{p-1\}, \emptyset), S, m - k)$. Note that B_1 is the set of $g \in G_p^{(n)}$ satisfying

$$\begin{aligned} |g^{-1}(x)|_n &> \frac{1}{p}|\Omega_n| + m && \text{for all } x \in S \\ |g^{-1}(\mathbb{F}_p \setminus \{x\})|_n &\geq \left(1 - \frac{1}{p}\right)|\Omega_n| + m && \text{for all } x \notin S. \end{aligned}$$

If g satisfies the above inequalities, then g satisfies the same inequalities with $m - k$ in place of m , so $g \in B'_1$.

Part (vi) follows from Parts (iv) and (v).

To prove Part (vii), first observe that $\{x\} \in \mathcal{S}_x$, by the definition of \mathcal{S}_x . Now

$$x\mathbf{1} \in \text{Bias}_n(\mathbb{F}_p, (\{0\}, \dots, \{p-1\}, \emptyset), \{x\}, m),$$

so $x\mathbf{1} \in P_x^{(n,m)}$. □

3.4. Iterating the partition. Let $l \in \mathbb{N}$, $l \geq 2$. Suppose that $\mathcal{P}^{(n_1, m_1), \dots, (n_{l-1}, m_{l-1})}$ is a partition of $G_p^{(n_{l-1})}$ defined for every increasing $l-1$ -tuple (n_1, \dots, n_{l-1}) and every $l-1$ -tuple (m_1, \dots, m_{l-1}) of natural numbers. As in Section 2.3, we identify elements of $G_p^{(n_l)}$ with functions $f : \Omega_{n_l} \rightarrow G_p^{(n_l - n_1)}$, using the identification

$$(3.1) \quad g \leftrightarrow (\tau \mapsto g|_\tau, \tau \in \Omega_{n_l}), \quad g \in G_p^{(n_l)}.$$

The set $\text{Bias}_{n_l}(G_p^{(n_l - n_1)}, \mathcal{P}^{(n_2 - n_1, m_2), \dots, (n_l - n_1, m_l)}, S, m_l)$ may be identified with a subset of $G_p^{(n_l)}$, using the identification (3.1).

Let (n_1, \dots, n_l) be an increasing l -tuple of natural numbers, and (m_1, \dots, m_l) an l -tuple of natural numbers. Define $\mathcal{P}^{(n_1, m_1), \dots, (n_l, m_l)}$, a partition of $G_p^{(n_l)}$, as follows. For each $x \in \mathbb{F}_p$,

$$P_x^{(n_1, m_1), \dots, (n_l, m_l)} := \bigcup_{S \in \mathcal{S}_x} \text{Bias}_{n_l}(G_p^{(n_l - n_1)}, \mathcal{P}^{(n_2 - n_1, m_2), \dots, (n_l - n_1, m_l)}, S, m_l)$$

$$Z^{(n_1, m_1), \dots, (n_l, m_l)} := G_p^{(n_l)} \setminus \bigcup_{x \in \mathbb{F}_p} P_x^{(n_1, m_1), \dots, (n_l, m_l)}.$$

Then

$$\mathcal{P}^{(n_1, m_1), \dots, (n_l, m_l)} := (P_0^{(n_1, m_1), \dots, (n_l, m_l)}, \dots, P_{p-1}^{(n_1, m_1), \dots, (n_l, m_l)}, Z^{(n_1, m_1), \dots, (n_l, m_l)}).$$

Note that $\text{Bias}_{n_l}(G_p^{(n_l - n_1)}, \mathcal{P}^{(n_2 - n_1, m_2), \dots, (n_l - n_1, m_l)}, S, m_l)$ is the set of $g \in G_p^{(n_l)}$ satisfying

$$|\{\tau \in \Omega_{n_l} : g|_\tau \in P_x^{(n_2 - n_1, m_2), \dots, (n_l - n_1, m_l)}\}| > \frac{1}{p}|\Omega_{n_l}| + m_1 \quad \text{for } x \in S$$

$$|\{\tau \in \Omega_{n_l} : g|_\tau \in P_x^{(n_2 - n_1, m_2), \dots, (n_l - n_1, m_l)}\}| < \frac{1}{p}|\Omega_{n_l}| - m_1 \quad \text{for } x \notin S.$$

Example. With $p = 3$, $\text{Bias}_{n_1}(G_3^{(n_2 - n_1)}, \mathcal{P}^{(n_2 - n_1, m_2)}, \{0\}, m_1)$ is the set of $g \in G_3^{(n_2)}$ satisfying

$$|\{\tau \in \Omega_{n_1} : g|_\tau \in P_0^{(n_2 - n_1, m_2)}\}| > \frac{1}{3}|\Omega_{n_1}| + m_1,$$

$$|\{\tau \in \Omega_{n_1} : g|_\tau \in P_y^{(n_2 - n_1, m_2)}\}| < \frac{1}{3}|\Omega_{n_1}| - m_1, y = 1, 2.$$

3.5. Notation. Expressions such as $(n_1, m_1), \dots, (n_l, m_l)$ will appear often. We abbreviate

$$\begin{aligned}\mathbf{n}_l &:= (n_1, m_1), \dots, (n_l, m_l), \\ \mathbf{n}'_l &:= (n_2 - n_1, m_2), \dots, (n_l - n_1, m_l).\end{aligned}$$

Note that \mathbf{n}_l has length l and \mathbf{n}'_l has length $l - 1$.

With these abbreviations, the definition of $P_x^{(n_1, m_1), \dots, (n_l, m_l)}$ becomes

$$P_x^{\mathbf{n}_l} := \bigcup_{S \in \mathcal{S}_x} \text{Bias}_{n_1}(G_p^{(n_l - n_1)}, \mathcal{P}^{\mathbf{n}'_l}, S, m_1).$$

3.6. Properties of the partitions. The following lemma summarizes the relevant properties of the partitions $\mathcal{P}^{\mathbf{n}_l}$.

Lemma 3.6. *Let $(n_i)_{i \in \mathbb{N}}$ be an increasing sequence of natural numbers, let $m_i, k_i \in \mathbb{N}$ satisfy $k_i < m_i < n_i$ for each i, j with $j < i$. Then for all $l \in \mathbb{N}$, all $x, y \in \mathbb{F}_p$, and all $j \in \{1, \dots, l\}$,*

- (i) $P_x^{\mathbf{n}_l} + y\mathbf{1} = P_{x+y}^{\mathbf{n}_l}$,
- (ii) if $u \in U(n_j, k_j)$, then

$$P_x^{\mathbf{n}_l} + u \subseteq P_x^{(n_1, m_1), \dots, (n_j, m_j - k_j), \dots, (n_l, m_l)},$$

- (iii) if $u \in U(n_j, k_j)$, then

$$P_x^{\mathbf{n}_l} + u + y\mathbf{1} \subseteq P_{x+y}^{(n_1, m_1), \dots, (n_j, m_j - k_j), \dots, (n_l, m_l)},$$

- (iv) $P_x^{\mathbf{n}_l} \cap P_y^{\mathbf{n}_l} = \emptyset$ if $x \neq y$,
- (v) $P_x^{\mathbf{n}_l} \subseteq P_x^{(n_1, m_1), \dots, (n_j, m_j - k_j), \dots, (n_l, m_l)}$,
- (vi) if $x \neq y$, then

$$\begin{aligned}P_x &:= P_x^{\mathbf{n}_l} \\ P'_y &:= P_y^{(n_1, m_1), \dots, (n_j, m_j - k_j), \dots, (n_l, m_l)}\end{aligned}$$

are mutually disjoint,

- (vii) If $\mathbf{n}_{l+1} = (n_1, m_1), \dots, (n_{l+1}, m_{l+1})$, then $P_x^{\mathbf{n}_l} \subseteq P_x^{\mathbf{n}_{l+1}}$.

Note: Part (i) implies $|P_x^{\mathbf{n}_l}| = |P_y^{\mathbf{n}_l}|$ for all $x, y \in \mathbb{F}_p$.

When we prove statements by induction on l , the induction hypothesis will refer to all sequences $((n_1, m_1), \dots, (n_{l-1}, m_{l-1}))$, where $n_i, m_i \in \mathbb{N}$ and (n_1, \dots, n_{l-1}) is increasing. In particular, the induction hypothesis for $l \geq 2$ will state that each of Parts (i)-(vii) hold with $\mathbf{n}'_l = (n_2 - n_1, m_2), \dots, (n_l - n_1, m_l)$ in place of $\mathbf{n}_l = (n_1, m_1), \dots, (n_l, m_l)$.

Proof. Part (i). Induction on l . The case $l = 1$ is Part (i) of Lemma 3.5. For $l > 1$, let $g \in P_x^{\mathbf{n}_l}$, so that

$$g \in \text{Bias}_{n_1}(G_p^{(n_l-n_1)}, \mathcal{P}^{\mathbf{n}'_l}, S, m_1)$$

for some $S \in \mathcal{S}_x$. For each $w \in \mathbb{F}_p$ and each $\tau \in \Omega_{n_1}$, the induction hypothesis implies $g|_\tau + y\mathbf{1}|_\tau \in P_{w+y}^{\mathbf{n}'_l}$ if and only if $g|_\tau \in P_w^{\mathbf{n}'_l}$. It follows that

$$g + y\mathbf{1} \in \text{Bias}_{n_1}(G_p^{(n_l-n_1)}, \mathcal{P}^{\mathbf{n}'_l}, S + y, m_1),$$

so that $g + y\mathbf{1} \in P_{x+y}^{\mathbf{n}_l}$.

We have proved that $P_x^{\mathbf{n}_l} + y\mathbf{1} \subseteq P_{x+y}^{\mathbf{n}_l}$, and the reverse containment follows by replacing x with $x + y$ and y with $-y$.

Part (ii). We consider three cases.

Case 1. $j = 1, l = 1$. This is Part (ii) of Lemma 3.5.

Case 2. $j = 1, l > 1$. Note that it suffices to show that for $S \in \mathcal{S}_x$ and $u \in U(n_1, k_1)$,

$$(3.2) \quad \begin{aligned} & \text{Bias}_{n_1}(G_p^{(n_l-n_1)}, \mathcal{P}^{\mathbf{n}'_l}, S, m_1) + u \\ & \subseteq \text{Bias}_{n_1}(G_p^{(n_l-n_1)}, \mathcal{P}^{\mathbf{n}'_l}, S, m_1 - k_1). \end{aligned}$$

Fix $g \in \text{Bias}_{n_1}(G_p^{(n_l-n_1)}, \mathcal{P}^{\mathbf{n}'_l}, S, m_1)$ and $w \in S$. We have $g|_\tau \in P_w^{\mathbf{n}'_l}$ for $> \frac{1}{p}|\Omega_{n_1}| + m_1$ values of $\tau \in \Omega_{n_1}$, and $g|_\tau + u|_\tau = g|_\tau$ for all but k_1 values of τ , so $g|_\tau + u|_\tau \in P_w^{\mathbf{n}'_l}$ for $> \frac{1}{p}|\Omega_{n_1}| + m_1 - k_1$ values of $\tau \in \Omega_{n_1}$.

Similarly, for $w \notin S$, we have $g|_\tau + u|_\tau \in P_w^{\mathbf{n}'_l}$ for $< \frac{1}{p}|\Omega_{n_1}| - m_1 + k_1$ values of $\tau \in \Omega_{n_1}$, establishing the containment (3.2).

Case 3. $j > 1, l > 1$. Induction on l . It suffices to show that for all $S \subseteq \mathbb{F}_p$ and $u \in U(n_j, k_j)$,

$$(3.3) \quad \begin{aligned} & \text{Bias}_{n_1}(G_p^{(n_l-n_1)}, \mathcal{P}^{\mathbf{n}'_l}, S, m_1) + u \subseteq \\ & \text{Bias}_{n_1}(G_p^{(n_l-n_1)}, \mathcal{P}^{(n_2-n_1, m_2), \dots, (n_j-n_1, m_j-k_j), \dots, (n_l-n_1, m_l)}, S, m_1). \end{aligned}$$

Let $S \subseteq \mathbb{F}_p$, $g \in \text{Bias}_{n_1}(G_p^{(n_l-n_1)}, \mathcal{P}^{\mathbf{n}'_l}, S, m_1)$ and $u \in U(n_j, k_j)$. For $w \in S$, we have $g|_\tau \in P_w^{\mathbf{n}'_l}$ for $> \frac{1}{p}|\Omega_{n_1}| + m_1$ values of $\tau \in \Omega_{n_1}$. For each $\tau \in \Omega_{n_1}$, we have $u|_\tau \in U(n_j - n_1, k_j)$. If $g|_\tau \in P_w^{\mathbf{n}'_l}$, then $g|_\tau + u|_\tau \in P_w^{(n_2-n_1, m_2), \dots, (n_j-n_1, m_j-k_j), \dots, (n_l-n_1, m_l)}$, by the induction hypothesis or by Case 1 or Case 2. Thus,

$$g|_\tau + u|_\tau \in P_w^{(n_2-n_1, m_2), \dots, (n_j-n_1, m_j-k_j), \dots, (n_l-n_1, m_l)}$$

for $> \frac{1}{p}|\Omega_{n_1}| + m_1$ values of $\tau \in \Omega_{n_1}$. Similarly, for $w \notin S$, we have $g|_\tau + u|_\tau \in P_w^{(n_2-n_1, m_2), \dots, (n_j-n_1, m_j-k_j), \dots, (n_l-n_1, m_l)}$ for $< \frac{1}{p}|\Omega_{n_1}| - m_1$ values

of $\tau \in \Omega_{n_1}$. This establishes the inclusion (3.3) and completes the induction and the proof of Part (ii).

Part (iii). This follows from Parts (i) and (ii).

Part (iv). Let $S, S' \subseteq \mathbb{F}_p$, $S \neq S'$. It suffices to show that the following sets are disjoint:

$$\begin{aligned} B &:= \text{Bias}_{n_1}(G_p^{(n_l-n_1)}, \mathcal{P}^{\mathbf{n}'_l}, S, m_1), \\ B' &:= \text{Bias}_{n_1}(G_p^{(n_l-n_1)}, \mathcal{P}^{\mathbf{n}'_l}, S', m_1). \end{aligned}$$

The desired disjointness follows from Lemma 3.1.

Part (v). Induction on l . The base case $l = 1$ is Part (v) of Lemma 3.5.

For $l > 1$, $j = 1$, it suffices to show that for each $S \subseteq \mathbb{F}_p$

$$B_2 := \text{Bias}_{n_1}(G_p^{(n_l-n_1)}, \mathcal{P}^{\mathbf{n}'_l}, S, m_1)$$

is contained in

$$B'_2 := \text{Bias}_{n_1}(G_p^{(n_l-n_1)}, \mathcal{P}^{\mathbf{n}'_l}, S, m_1 - k_1).$$

Let $g \in B_2$. For $w \in \mathbb{F}_p$, let $C_w^{\mathbf{n}'_l} := \bigcup_{x \neq w} P_x^{\mathbf{n}'_l}$. Then (using Part (iv))

$$(3.4) \quad |\{\tau \in \Omega_{n_1} : g|_\tau \in P_w^{\mathbf{n}'_l}\}| > \frac{1}{p}|\Omega_{n_1}| + m_1 \quad \text{if } w \in S,$$

$$(3.5) \quad |\{\tau \in \Omega_{n_1} : g|_\tau \in C_w^{\mathbf{n}'_l}\}| \geq \left(1 - \frac{1}{p}\right)|\Omega_{n_1}| + m_1 \quad \text{if } w \notin S.$$

If g satisfies these inequalities, then it satisfies the same inequalities with $m_1 - k_1$ in place of m_1 , so $g \in B'_2$.

For $l > 1$, $j > 1$, it suffices to show that for each $S \subseteq \mathbb{F}_p$

$$B_3 := \text{Bias}_{n_1}(G_p^{(n_l-n_1)}, \mathcal{P}^{(n_2-n_1, m_2), \dots, (n_j-n_1, m_j), \dots, (n_l-n_1, m_l)}, S, m_1)$$

is contained in

$$B'_3 := \text{Bias}_{n_1}(G_p^{(n_l-n_1)}, \mathcal{P}^{(n_2-n_1, m_2), \dots, (n_j-n_1, m_j-k_j), \dots, (n_l-n_1, m_l)}, S, m_1).$$

For $w \in \mathbb{F}_p$, let $P_w := P_w^{(n_2-n_1, m_1), \dots, (n_j-n_1, m_j), \dots, (n_l-n_1, m_l)}$, and let $C_w := \bigcup_{x \neq w} P_x$. Note that $g \in B_3$ if and only if the following inequalities hold:

$$(3.6) \quad |\{\tau \in \Omega_{n_1} : g|_\tau \in P_w\}| > \frac{1}{p}|\Omega_n| + m_1 \quad \text{if } w \in S,$$

$$(3.7) \quad |\{\tau \in \Omega_{n_1} : g|_\tau \in C_w\}| \geq \left(1 - \frac{1}{p}\right)|\Omega_n| + m_1 \quad \text{if } w \notin S.$$

By the induction hypothesis,

$$P_w \subseteq P'_w := P_w^{(n_2-n_1, m_1), \dots, (n_j-n_1, m_j-k_j), \dots, (n_l-n_1, m_l)},$$

and consequently $C_w \subseteq C'_w := \bigcup_{x \neq w} P'_x$, so if g satisfies inequalities (3.6) and (3.7), then g satisfies the same inequalities with P'_w and C'_w in place of P_w and C_w , which implies $g \in B'_3$.

Part (vi). By Part (v), $P_x^{\mathbf{n}_l} \subseteq P_x^{(n_1, m_1), \dots, (n_j, m_j - k_j), \dots, (n_l, m_l)} =: P'_x$. By Part (iv), $P'_x \cap P'_y = \emptyset$, so $P_x \cap P'_y = \emptyset$.

Part (vii). Induction on l . For $l = 1$ we claim that for all $x \in \mathbb{F}_p$, $P_x^{(n_1, m_1)} \subseteq P_x^{(n_1, m_1), (n_2, m_2)}$. To prove this, let $g \in P_x^{(n_1, m_1)}$, so that $g \in \text{Bias}_{n_1}(\mathbb{F}_p, (\{0\}, \dots, \{p-1\}, \emptyset), S, m_1)$ for some $S \in \mathcal{S}_x$. We will show that

$$(3.8) \quad g \in \text{Bias}_{n_1}(G_p^{(n_2 - n_1)}, \mathcal{P}^{(n_2 - n_1, m_2)}, S, m_1).$$

Recall that the cells of $\mathcal{P}^{(n_2 - n_1, m_2)}$ are the $P_x^{(n_2 - n_1, m_2)}$ and $Z^{(n_2 - n_1, m_2)}$. For each $\tau \in \Omega_{n_1}$, we have $g|_\tau = y\mathbf{1}$ for some $y \in \mathbb{F}_p$, and $y\mathbf{1} \in P_y^{(n_2 - n_1, m_2)}$, by Part (vii) of Lemma 3.5. From the definition of $\text{Bias}_n(\cdot)$ we then conclude that the inclusion (3.8) holds. This suffices to prove that $g \in P_x^{(n_1, m_1), (n_2, m_2)}$, concluding the case $l = 1$ of the induction.

Now fix $l \in \mathbb{N}$, $l > 1$, and $g \in P_x^{\mathbf{n}_l}$. We must prove that $g \in P_x^{(\mathbf{n}_{l+1})}$. Let $S \in \mathcal{S}_x$ be such that

$$(3.9) \quad g \in \text{Bias}_{n_1}(G_p^{(n_l - n_1)}, \mathcal{P}^{(n_2 - n_1, m_2), \dots, (n_l - n_1, m_l)}, S, m_1).$$

We will show that

$$(3.10) \quad g \in \text{Bias}_{n_1}(G_p^{(n_{l+1} - n_1)}, \mathcal{P}^{(n_2 - n_1, m_2), \dots, (n_{l+1} - n_1, m_{l+1})}, S, m_1).$$

The induction hypothesis implies that for each $\tau \in \Omega_{n_1}$ and $y \in \mathbb{F}_p$, if $g|_\tau \in P_y^{\mathbf{n}'_l}$, then $g|_\tau \in P_y^{(n_2 - n_1, m_1), \dots, (n_{l+1} - n_1, m_{l+1})}$. Inclusion (3.9) then implies Inclusion (3.10), and we conclude $g \in P_x^{(\mathbf{n}_{l+1})}$. This completes the induction and the proof of Part (vii). \square

3.7. Estimating $|P_x^{\mathbf{n}_l}|$. The next lemma estimates the cardinality of $P_x^{(n, m)}$ for large n and fixed m .

Lemma 3.7. Fix $m \in \mathbb{N}$. For all $x \in \mathbb{F}_p$, $\lim_{n \rightarrow \infty} \frac{|P_x^{(n, m)}|}{|G_p^{(n)}|} = \frac{1}{p}$.

Remark 3.8. We can write the conclusion of Lemma 3.7 as

$$(3.11) \quad |P_x^{(n, m)}| = |G_p^{(n)}| \left(\frac{1}{p} + o(1) \right),$$

where $o(1)$ is a quantity that tends to 0 as $n \rightarrow \infty$, for fixed m .

Proof. It suffices to show that $\lim_{n \rightarrow \infty} \frac{|Z^{(n, m)}|}{|G_p^{(n)}|} = 0$, since $|P_x^{(n, m)}| = |P_0^{(n, m)}|$ for each $x \in \mathbb{F}_p$ (by Part (i) of Lemma 3.5), and the sets $P_x^{(n, m)}$, $x \in \mathbb{F}_p$, partition $G_p^{(n)} \setminus Z^{(n, m)}$.

Now $Z^{(n,m)}$ is the set of $g \in G_p^{(n)}$ such that

$$\frac{1}{p}|\Omega_n| - m \leq |g^{-1}(y)|_n \leq \frac{1}{p}|\Omega_n| + m \quad \text{for some } y \in \mathbb{F}_p.$$

Let

$$M_{n,m} := \max \left\{ \binom{|\Omega_n|}{t} : \frac{1}{p}|\Omega_n| - m \leq t \leq \frac{1}{p}|\Omega_n| + m \right\}.$$

If t satisfies $\frac{1}{p}|\Omega_n| - m \leq t \leq \frac{1}{p}|\Omega_n| + m$ and $y \in \mathbb{F}_p$, the number of $g \in G_p^{(n)}$ satisfying $g^{-1}(y) = t$ is at most

$$(p-1)^{|\Omega_n| - \lfloor \frac{1}{p}|\Omega_n| - m \rfloor} M_{n,m}.$$

Summing over all possible values of t and y , we get

$$|Z^{(n,m)}| \leq p(2m+1)(p-1)^{|\Omega_n| - \lfloor \frac{1}{p}|\Omega_n| - m \rfloor} M_{n,m}.$$

Estimating the binomial coefficients in the definition of $M_{n,m}$ with Stirling's formula, we find $|Z^{(n,m)}| = o(p^{|\Omega_n|}) = o(|G_p^{(n)}|)$. \square

In order to estimate the cardinality of $P_x^{\mathbf{n}_l}$, we construct some elements of $P_x^{\mathbf{n}_l}$ from elements of $P_x^{\mathbf{n}_{l-1}}$.

Definition 3.9. For $n_{l-1} < n_l \in \mathbb{N}$ and $g \in G_p^{(n_{l-1})}$, let $G_p^{(n_l)}[g, m_l]$ be the set of $h \in G_p^{(n_l)}$ satisfying $h|_\tau \in P_{g(\tau)}^{(n_l - n_{l-1}, m_l)}$ for each $\tau \in \Omega_{n_{l-1}}$.

We abuse notation in the expression $P_{g(\tau)}^{(n_l - n_{l-1}, m_l)}$, writing $g(\tau)$ for the element $x \in \mathbb{F}_p$ satisfying $g([\tau]) = \{x\}$.

Lemma 3.10. Let $l \in \mathbb{N}$, $l \geq 2$. Let $\mathbf{n}_l = (n_1, m_1), \dots, (n_l, m_l)$ where $n_i, m_i \in \mathbb{N}$, n_i is increasing, and let $x \in \mathbb{F}_p$. Then

(i) If $g, g' \in G_p^{(n_{l-1})}$, $g \neq g'$, then

$$G_p^{(n_l)}[g, m_l] \cap G_p^{(n_l)}[g', m_l] = \emptyset.$$

(ii) If $g \in P_x^{(n_1, m_1), \dots, (n_{l-1}, m_{l-1})}$ then $G_p^{(n_l)}[g, m_l] \subseteq P_x^{\mathbf{n}_l}$.

(iii) If $g \in G_p^{(n_{l-1})}$ and m_l are fixed, then

$$(3.12) \quad |G_p^{(n_l)}[g, m_l]| = |G_p^{(n_l)}| \left(\frac{1}{|G_p^{(n_{l-1})}|} + o(1) \right),$$

where $o(1)$ is a quantity tending to 0 as $n_l \rightarrow \infty$.

Proof. Part (i) follows from the fact that the $P_x^{(n_l - n_{l-1}, m_l)}$, $x \in \mathbb{F}_p$, are mutually disjoint (Part (iv) of Lemma 3.6).

We prove Part (ii) by induction on l . For $l = 1$ there is nothing to prove. For $l = 2$, fix a $g \in P_x^{(n_1, m_1)}$, so that

$$g \in \text{Bias}_{n_1}(\mathbb{F}_p, (\{0\}, \dots, \{p-1\}, \emptyset), S, m_1)$$

for some $S \in \mathcal{S}_x$. Let $h \in G_p^{(n_2)}[g, m_2]$. For each $w \in S$, $|g^{-1}(w)|_n > \frac{1}{p}|\Omega_{n_1}| + m_1$, and for each $w \notin S$, $|g^{-1}(w)|_n < \frac{1}{p}|\Omega_{n_1}| - m_1$. Then

$$|\{\tau \in \Omega_{n_1} : h|_\tau \in P_w^{(n_2-n_1, m_2)}\}| \begin{cases} > \frac{1}{p}|\Omega_{n_1}| + m_1, & \text{if } w \in S; \\ < \frac{1}{p}|\Omega_{n_1}| - m_1, & \text{if } w \notin S. \end{cases}$$

Consequently $h \in \text{Bias}_{n_1}(G_p^{(n_2-n_1)}, \mathcal{P}^{(n_2-n_1, m_2)}, S, m_1)$, and we conclude that $h \in P_x^{(n_1, m_1), (n_2, m_2)}$.

For $l > 2$, write \mathbf{n}_{l-1} for $(n_1, m_1), \dots, (n_{l-1}, m_{l-1})$. Let $g \in P_x^{\mathbf{n}_{l-1}}$, so $g \in \text{Bias}_{n_1}(G_p^{(n_{l-1}-n_1)}, \mathcal{P}^{\mathbf{n}'_{l-1}}, S, m_1)$ for some $S \in \mathcal{S}_x$. Then

$$(3.13) \quad |\{\tau \in \Omega_{n_1} : g|_\tau \in P_w^{\mathbf{n}'_{l-1}}\}| > \frac{1}{p}|\Omega_{n_1}| + m_1, \text{ if } w \in S$$

$$(3.14) \quad |\{\tau \in \Omega_{n_1} : g|_\tau \in P_w^{\mathbf{n}'_{l-1}}\}| < \frac{1}{p}|\Omega_{n_1}| - m_1, \text{ if } w \notin S.$$

Let $h \in G_p^{(n_l)}[g, m_l]$. For each $\tau \in \Omega_{n_1}$, the definitions of h and $G_p^{(n_l)}[g, m_l]$ imply

$$(3.15) \quad h|_\tau \in G_p^{(n_l-n_1)}[g|_\tau, m_l].$$

By the induction hypothesis and the inclusion (3.15), we have $h|_\tau \in P_x^{\mathbf{n}'_l}$ whenever $g|_\tau \in P_x^{\mathbf{n}'_{l-1}}$. Then Inequalities (3.13) and (3.14) imply

$$h \in \text{Bias}_{n_1}(G_p^{(n_l-n_1)}, \mathcal{P}^{\mathbf{n}'_l}, S, m_1),$$

and we conclude that $h \in P_x^{\mathbf{n}_l}$.

To prove Part (iii), fix $g \in G_p^{(n_{l-1})}$. We identify $G_p^{(n_l)}[g, m_l]$ with the set of functions $h : \Omega_{n_{l-1}} \rightarrow G_p^{(n_l-n_{l-1})}$ satisfying $h(\tau) \in P_{g(\tau)}^{(n_l-n_1, m_l)}$ for each $\tau \in \Omega_{n_{l-1}}$. There are

$$\prod_{\tau \in \Omega_{n_{l-1}}} |P_{g(\tau)}^{(n_l-n_{l-1}, m_l)}|$$

such functions, and this quantity may be simplified using the estimate (3.11) to conclude

$$\begin{aligned}
|G_p^{(n_l)}[g, m_l]| &= \prod_{\tau \in \Omega_{n_{l-1}}} |G_p^{(n_l - n_{l-1})}| \left(\frac{1}{p} + o(1) \right) \\
&= |G_p^{(n_l - n_{l-1})}|^{|\Omega_{n_{l-1}}|} \left(\frac{1}{p} + o(1) \right)^{|\Omega_{n_{l-1}}|} \\
&= |G_p^{(n_l)}| \left(\frac{1}{p^{|\Omega_{n_{l-1}}|}} + o(1) \right) \\
&= |G_p^{(n_l)}| \left(\frac{1}{|G_p^{(n_{l-1})}|} + o(1) \right),
\end{aligned}$$

establishing the estimate (3.12). \square

Lemma 3.11. *Let $l \in \mathbb{N}$, $l > 1$. Fix natural numbers $n_1 < \dots < n_{l-1}$, m_1, \dots, m_l . Then for each $x \in \mathbb{F}_p$*

$$(3.16) \quad \liminf_{n_l \rightarrow \infty} \frac{|P_x^{(n_1, m_1), \dots, (n_l, m_l)}|}{|G_p^{(n_l)}|} \geq \frac{|P_x^{(n_1, m_1), \dots, (n_{l-1}, m_{l-1})}|}{|G_p^{(n_{l-1})}|}.$$

Proof. For $g \in G_p^{(n_{l-1})}$, let $B(g) := G_p^{(n_l)}[g, m_l]$. Lemma 3.10 implies that the $B(g)$ are mutually disjoint and that $P_x^{n_l} \supset \bigcup_{g \in P_x^{n_{l-1}}} B(g)$. We then have

$$\begin{aligned}
|P_x^{n_l}| &\geq |P_x^{n_{l-1}}| \min_g |B(g)| \\
&\geq |P_x^{n_{l-1}}| |G_p^{(n_l)}| \left(\frac{1}{|G_p^{(n_{l-1})}|} + o(1) \right) \quad \text{by Part (iii) of Lemma 3.10} \\
&= |G_p^{(n_l)}| \left(\frac{|P_x^{n_{l-1}}|}{|G_p^{(n_{l-1})}|} + o(1) \right),
\end{aligned}$$

which implies Inequality (3.16). \square

4. PROOF OF THEOREMS 1.2 AND 1.4

We maintain the notation and conventions of Sections 2 and 3.

Lemma 4.1. *Let $(n_j)_{j \in \mathbb{N}}$, $(m_j)_{j \in \mathbb{N}}$, $(k_j)_{j \in \mathbb{N}}$ be sequences of natural numbers with n_j increasing and $k_j < m_j$. Fix $x \in \mathbb{F}_p$ and let $A := \bigcup_{l=1}^{\infty} P_x^{n_l}$, $S := \bigcup_{j=1}^{\infty} V(n_j, k_j)$, where $V(n_j, k_j) = U(n_j, k_j) + \mathbf{1}$. Then*

$$A + S \subseteq \bigcup_{l=1}^{\infty} P_{x+1}^{(n_1, m_1 - k_1), \dots, (n_l, m_l - k_l)}.$$

Proof. We claim that for all $j, l \in \mathbb{N}$,

$$(4.1) \quad P_x^{n_l} + V(n_j, k_j) \subseteq P_{x+1}^{(n_1, m_1 - k_1), \dots, (n_r, m_r - k_r)}$$

where $r = \max(j, l)$. The containment (4.1) follows from Parts (vii), (v), and (iii) of Lemma 3.6. Since $A + S = \bigcup_{j,l=1}^{\infty} P_x^{n_l} + V(n_j, k_j)$, the containment (4.1) implies the conclusion. \square

Proof of Theorem 1.2, Part 1 and Theorem 1.4. We will find a set $A \subseteq G_p$ such that $d^*(A) > \delta_p - \varepsilon$ and a set $S \subseteq G_p$ dense in the Bohr topology of G_p such that $(A - A) \cap S = \emptyset$. Note that this suffices to prove Part 1 of Theorem 1.2, as every set containing a Bohr neighborhood has nonempty intersection with S .

Fix a prime number p . Let $(k_i)_{i \in \mathbb{N}}$ be an increasing sequence of natural numbers. Let $m_i = 3k_i$, and apply Lemmas 3.7 and 3.11 to find an increasing sequence $(n_i)_{i \in \mathbb{N}}$ satisfying

$$(4.2) \quad \frac{|P_x^{(n_1, m_1), \dots, (n_l, m_l)}|}{|G_p^{(n_l)}|} > \frac{1 - \varepsilon}{p} \quad \text{for all } l \in \mathbb{N}, x \in \mathbb{F}_p.$$

Let $E \subseteq \mathbb{F}_p$, $E = \{1, 3, \dots, p-1\}$, so that $(E+1) \cap E = \emptyset$ and

$$(4.3) \quad \frac{|E|}{p} = \begin{cases} \frac{1}{2}, & \text{if } p = 2; \\ \frac{1}{2} - \frac{1}{2p}, & \text{if } p \text{ is odd.} \end{cases}$$

Let $S = \bigcup_{i \in \mathbb{N}} V(n_i, k_i)$, so that S is dense in the Bohr topology of G_p by Lemma 2.6.

Let $A_x := \bigcup_{l \in \mathbb{N}} P_x^{(n_1, m_1), \dots, (n_l, m_l)}$, and let $A := \bigcup_{x \in E} A_x$. We claim that

$$(4.4) \quad d^*(A) > \delta_p - \varepsilon, \text{ and}$$

$$(4.5) \quad (A + S) \cap A = \emptyset.$$

Equation (4.5) implies $(A - A) \cap S = \emptyset$. Since S is dense in the Bohr topology of G_p , Inequality (4.4) and Equation (4.5) together imply Part 1 of Theorem 1.2. To prove Inequality (4.4), note that $\frac{|A_x \cap G_p^{(n_l)}|}{|G_p^{(n_l)}|} > \frac{1 - \varepsilon}{p}$ for each $l \in \mathbb{N}$, $x \in \mathbb{F}_p$, by Inequality (4.2). The mutual disjointness of $P_x^{(n_1, m_1), \dots, (n_l, m_l)}$, $x \in \mathbb{F}_p$, then implies $\frac{|A \cap G_p^{(n_l)}|}{|G_p^{(n_l)}|} > \frac{|E|}{p} - \varepsilon$, and the last estimate combined with Inequality (2.2) implies Inequality (4.4).

To prove Equation (4.5), apply Lemma 4.1 to obtain

$$(4.6) \quad A_x + S \subseteq P'_{x+1} := \bigcup_{l \in \mathbb{N}} P_{x+1}^{(n_1, m_1 - k_1), \dots, (n_l, m_l - k_l)}$$

for each $x \in E$. Parts (v) and (vi) of Lemma 3.6 and the definition of E imply P'_{x+1} is disjoint from A_y for each $y \in E$, so $(A_x + S) \cap A_y = \emptyset$ for all $x, y \in E$. We conclude that $(A + S) \cap A = \emptyset$, completing the proof of Theorem 1.2, Part 1.

Setting $A' := A + S$, the containment (4.6) and the argument demonstrating $(A - A) \cap S = \emptyset$ also implies $(A' - A') \cap S = \emptyset$. Consequently $A' - A'$ does not contain a Bohr neighborhood, so Observation 2.8 implies that A' is not piecewise Bohr. Since $d^*(A') \geq d^*(A) > \delta_p - \varepsilon$ and S is dense in the Bohr topology of G_p , this completes the proof of Theorem 1.4. \square

We need the following lemma to prove Part 2 of Theorem 1.2.

Lemma 4.2. *If G is a countable abelian group, $F \subseteq G$ is finite, and $A \subseteq G$, then there is a $g \in G$ such that $|(A - g) \cap F| \geq d^*(A)|F|$.*

Proof. Let $\Phi = (\Phi_j)_{j \in \mathbb{N}}$ be a Følner sequence such that $d^*(A) = d_\Phi(A)$. Let $\varepsilon, \varepsilon' > 0$ and $F \subseteq G$ a finite set. For sufficiently large j , we have

$$\begin{aligned} \frac{1}{|\Phi_j|} \sum_{g \in \Phi_j} |(A - g) \cap F| &= \frac{1}{|\Phi_j|} \sum_{g' \in F} \sum_{g \in \Phi_j} 1_{A-g}(g') \\ &= \frac{1}{|\Phi_j|} \sum_{g' \in F} \sum_{g \in \Phi_j} 1_A(g + g') \\ &= \sum_{g' \in F} \frac{1}{|\Phi_j|} |A \cap (\Phi_j + g')| \\ &\geq \sum_{g' \in F} \frac{1}{|\Phi_j|} (|A \cap \Phi_j| - \varepsilon). \end{aligned}$$

For ε sufficiently small, the above inequalities imply that

$$\frac{1}{|\Phi_j|} \sum_{g \in \Phi_j} |(A - g) \cap F| \geq d^*(A)|F| - \varepsilon' \quad \text{for sufficiently large } j.$$

The pigeonhole principle then implies $|(A - g) \cap F| \geq d^*(A)|F| - \varepsilon'$ for some $g \in G$. Using the integrality of $|(A - g) \cap F|$, we conclude that $|(A - g) \cap F| \geq d^*(A)|F|$ for some g . \square

Lemma 4.3. *Let p be prime. If $A \subseteq G_p$ has $d^*(A) > \delta_p$, then $A - A = G_p$.*

Proof. Let $h \in G_p$, $h \neq 0$ and let $H := \{0, h, 2h, \dots, (p-1)h\}$ be the subgroup generated by h . Then $|H| = p$. By Lemma 4.2, there is a $g \in G$ such that $|(A - g) \cap H| \geq d^*(A)|H|$. For this g , the inequality

$d^*(A)|H| > \delta_p|H|$ implies $|(A-g) \cap H|$ is an integer greater than $\delta_p|H|$, and we conclude that $|(A-g) \cap H| > \frac{1}{2}|H|$. Since H is a subgroup, the last inequality implies $[(A-g) \cap H] - [(A-g) \cap H] = H$. We conclude that $h \in (A-g) - (A-g) = A - A$. Since h was an arbitrary nonzero element of G_p , we have shown that $A - A = G_p$. \square

Part 2 of Theorem 1.2 now follows from Lemma 4.3.

REFERENCES

1. Vitaly Bergelson, Hillel Furstenberg, and Benjamin Weiss, *Piecewise-Bohr sets of integers and combinatorial number theory*, Topics in discrete mathematics, Algorithms Combin., vol. 26, Springer, Berlin, 2006, pp. 13–37.
2. Vitaly Bergelson and Imre Z. Ruzsa, *Sumsets in difference sets*, Israel J. Math. **174** (2009), 1–18.
3. N.N. Bogolyubov, *Some algebraical properties of almost periods*, Zap. Kafedry Mat. Fiz. Kiev **4** (1939), 185–194.
4. Erling Følner, *Generalization of a theorem of Bogoliouboff to topological abelian groups. With an appendix on Banach mean values in non-abelian groups*, Math. Scand. **2** (1954), 5–18.
5. Alan Hunter Forrest, *Recurrence in dynamical systems: A combinatorial approach*, ProQuest LLC, Ann Arbor, MI, 1990, Thesis (Ph.D.)—The Ohio State University.
6. Alfred Geroldinger and Imre Z. Ruzsa, *Combinatorial number theory and additive group theory*, Advanced Courses in Mathematics. CRM Barcelona, Birkhäuser Verlag, Basel, 2009, Courses and seminars from the DocCourse in Combinatorics and Geometry held in Barcelona, 2008.
7. John T. Griesmer, *Sumsets of dense sets and sparse sets*, Israel J. Math. **190** (2012), 229–252.
8. ———, *Single recurrence in abelian groups*, (2017).
9. Norbert Hegyvári and Imre Z. Ruzsa, *Additive structure of difference sets and a theorem of Følner*, Australas. J. Combin. **64** (2016), 437–443.
10. Igor Kříž, *Large independent sets in shift-invariant graphs: solution of Bergelson’s problem*, Graphs Combin. **3** (1987), no. 2, 145–158.
11. Walter Rudin, *Fourier analysis on groups*, Interscience Tracts in Pure and Applied Mathematics, No. 12, Interscience Publishers (a division of John Wiley and Sons), New York-London, 1962.

DEPARTMENT OF APPLIED MATHEMATICS AND STATISTICS, COLORADO SCHOOL OF MINES, GOLDEN, COLORADO

E-mail address: jtgriesmer@gmail.com